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Coordinate Stretching and Interface Location II. A New PL Expansion[1]

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ABSTRACT

An alternative to the perturbation expansion of Poincaré and Lighthill is proposed, and a simple method of deriving the new expansion is given, for the case of a non-singular boundary. The new formulation has the advantage that the process of stretching of the independent coordinate is uncoupled from the rest of the problem, which can then be handled by standard techniques and programs. The method is illustrated by a simple example of pulsational instability in a one-zone approximation to a variable star.

I. INTRODUCTION

The method of coordinate stretching in perturbation analysis was initiated by Lindstedt [1], Poincaré [2], and Lighthill [3], and consists of an expansion of independent as well as dependent variables. In an earlier paper (Paper I, Usher [4]) we applied the PL perturbation

$$y^i = y_0^i + y_1^i + \cdots \qquad (i = 1, 2, 3, 4),$$
$$x = x_0 + x_1 + \cdots$$

to the system

$$dy^i/dx = f^i(y, x)$$

governing the structure of a star in quasi-static equilibrium. The resulting perturbation equations in first order were found to contain the arbitrary function x_1 and its derivative (with respect to x_0) in the nonhomogeneous terms, so that a choice

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for x_1 is necessary before the complete solution to the differential equations can be found. In this paper we derive an alternative to the conventional PL expansion which contains the arbitrary function in the expansion for y and not in the differential equations. In fact, the system of differential equations is the same as that for which no PL expansion has been applied; thus standard methods can be used for their solution in both exact and numerical treatments and the method of coordinate stretching can be used as a second step if necessary.

In Section II the new PL expansion is derived in the context of the small-parameter method, although in some problems which are essentially nonlinear it is convenient to set this small parameter equal to unity [5]. In addition we consider for the present only those cases for which the boundary points are nonsingular. In Section III we present a method of derivation which is useful because of its simplicity, and in Section IV we illustrate the method by a simple example.

II. Equations

A. *Perturbation Equations*

We consider the system of nonlinear equations

$$dv/du = F(u, v, \epsilon) \qquad (\epsilon \ll 1), \tag{1}$$

where v and F are n-component vectors. We assume that F is analytic in ϵ and can be written as a power series in ϵ:

$$dv/du = f(u, v) + \epsilon g(u, v) + \epsilon^2 h(u, v) + \cdots; \tag{2}$$

correspondence between these equations and those of Paper I is found by setting $g, h, ... \equiv 0$. We expand the dependent variables v and the independent-variable u in the manner usually employed (Poincare [1], Lighthill [2], Tsien [6], Krook [5]):

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots, \tag{3}$$

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots. \tag{4}$$

Here u_0 is the new independent variable and all other terms are functions of u_0. Throughout this paper subscripts on a function denote the order of the function and superscripts on a vector denote the component of that vector; i.e., v_2^i denotes the ith component of the second-order vector function v_2. For simplicity we consider terms up to and including second order only.

Taking derivatives with respect to u_0 rather than u, we rewrite Eq. (2) as

$$\frac{dv}{du_0} = \frac{du}{du_0}\left[f(u, v) + \epsilon g(u, v) + \epsilon^2 h(u, v) + \cdots\right], \tag{5}$$

where

$$\frac{dv}{du_0} = \frac{dv_0}{du_0} + \epsilon\frac{dv_1}{du_0} + \epsilon^2\frac{dv_2}{du_0} + \cdots \tag{6}$$

and

$$\frac{du}{du_0} = 1 + \epsilon\frac{du_1}{du_0} + \epsilon^2\frac{du_2}{du_0} + \cdots \tag{7}$$

from Eqs. (3) and (4). We expand the functions f, g, h, ... about the values (v_0, u_0) using Taylor's theorem; thus, summing over repeated indices we have for f,

$$\begin{aligned} f(u, v) = f(u_0, v_0) &+ \epsilon\left[v_1{}^i\left(\frac{\partial f}{\partial v^i}\right)_0 + u_1\left(\frac{\partial f}{\partial u}\right)_0\right] \\ &+ \epsilon^2\left[v_2{}^i\left(\frac{\partial f}{\partial v^i}\right)_0 + u_2\left(\frac{\partial f}{\partial u}\right)_0 + \frac{1}{2}v_1{}^i v_1{}^j\left(\frac{\partial^2 f}{\partial v^i \partial v^j}\right)_0\right. \\ &\left. + v_1{}^i u_1\left(\frac{\partial^2 f}{\partial v^i \partial u}\right)_0 + \frac{1}{2}u_1{}^2\left(\frac{\partial^2 f}{\partial u^2}\right)_0\right] + \cdots. \end{aligned} \tag{8}$$

Similar expressions hold for $g(u, v)$ and $h(u, v)$; quantities with zero subscripts are evaluated at (u_0, v_0).

We substitute Eqs. (6)–(8) in (5) and separate terms of equal order in ϵ and obtain the standard PL-perturbation equations up to order 2, as follows.

$$\epsilon^0: \quad \frac{dv_0}{du_0} = f_0; \tag{9}$$

$$\epsilon^1: \quad \frac{dv_1}{du_0} = v_1{}^i\left(\frac{\partial f}{\partial v^i}\right)_0 + u_1\left(\frac{\partial f}{\partial u}\right)_0 + f_0\frac{du_1}{du_0} + g_0; \tag{10}$$

$$\begin{aligned} \epsilon^2: \quad \frac{dv_2}{du_0} = {} & v_2{}^i\left(\frac{\partial f}{\partial v^i}\right)_0 + u_2\left(\frac{\partial f}{\partial u}\right)_0 + f_0\frac{du_2}{du_0} + h_0 \\ & + \frac{1}{2}v_1{}^i v_1{}^j\left(\frac{\partial^2 f}{\partial v^i \partial v^j}\right)_0 + u_1 v_1{}^i\left(\frac{\partial^2 f}{\partial v^i \partial u}\right)_0 + \frac{1}{2}u_1{}^2\left(\frac{\partial^2 f}{\partial u^2}\right)_0 \\ & + v_1{}^i\left(\frac{\partial g}{\partial v^i}\right)_0 + u_1\left(\frac{\partial g}{\partial u}\right)_0 \\ & + \frac{du_1}{du_0}\left[v_1{}^i\left(\frac{\partial f}{\partial v^i}\right)_0 + u_1\left(\frac{\partial f}{\partial u}\right)_0 + g_0\right]. \end{aligned} \tag{11}$$

The conventional (non-PL) perturbation equations are recovered by letting

$$u_0 = u, \tag{12}$$

$$u_i \equiv 0, (i = 1, 2, ...); \tag{13}$$

thus

$$\epsilon^0: \quad \frac{dv_0}{du} = f_0; \tag{14}$$

$$\epsilon^1: \quad \frac{dv_1}{du} = v_1{}^i \left(\frac{\partial f}{\partial v^i}\right)_0 + g_0; \tag{15}$$

$$\epsilon^2: \quad \frac{dv_2}{du} = v_2{}^i \left(\frac{\partial f}{\partial v^i}\right)_0 + h_0 + \frac{1}{2} v_1{}^i v_1{}^j \left(\frac{\partial^2 f}{\partial v^i \partial v^j}\right)_0 + v_1{}^i \left(\frac{\partial g}{\partial v^i}\right)_0 \tag{16}$$

B. *First-Order Equations and Boundary Conditions*

The essence of the PL method is that $u_i(u_0)$, $(i = 1, 2, ...)$ are arbitrary and can be chosen to facilitate the solution of the problem at hand. For the first-order equation (10), we customarily choose a relation between the terms in u_1 occurring in that equation.

Quite generally, let us suppose (see, e.g., Wasow [7]) that

$$f_0 \frac{du_1}{du_0} + u_1 \left(\frac{\partial f}{\partial u}\right)_0 = G_0(u_0), \tag{17}$$

where G_0 is an n-component vector, one component of which (say, $G_0{}^k$) is arbitrary. Having specified this component, the function u_1 can be found and Eq. (17) determines the remaining $(n - 1)$ components of G_0. The first-order PL equation (10) then becomes

$$\frac{dv_1}{du_0} = v_1{}^i \left(\frac{\partial f}{\partial v^i}\right)_0 + g_0 + G_0. \tag{18}$$

In applications of the PL method, Eqs. (17) and (18) along with appropriate boundary conditions can be solved once the arbitrary component $G_0{}^k$ is specified. However, at an early stage of the solution it is not always obvious which component of G_0 should be arbitrary and what the form of that component should be. Indeed, it is often not apparent whether the PL expansion is necessary at all. Consequently, it is desirable to postpone the choosing of $G_0{}^k$ until as much of the analysis as possible is completed.

According to available methods such a postponement would mean solving the non-PL equations (14)–(16) first and, on establishing the need for the PL expansion, to return to the PL equations (9)–(11) with knowledge of the difficulties involved (for example, the series solution may break down in some part of the domain of interest). The arbitrary component $G_0{}^k$ would then have to be decided upon, by use of the criterion that some or all of the difficulties of solution be eliminated. This necessitates solving the kth component of Eq. (17) with the correct choice for $G_0{}^k$ such that the resulting expressions for v_1 and u_1 overcome the difficulties of solution. Only in the easiest cases can a satisfactory choice be made simply by inspection. We shall show that the procedure described above can be made more direct and more efficient by foregoing the separation of Eq. (10) into Eqs. (17) and (18) by means of the function G_0, and that the new approach enables the arbitrary PL functions to be chosen easily by inspection of algebraic equations rather than by finding particular solutions of differential equations.

The essential feature of the development in first order is to note that

$$\frac{d}{du_0}(u_1 f_0) - u_1 f_0{}^i \left(\frac{\partial f}{\partial v^i}\right)_0 = f_0 \frac{du_1}{du_0} + u_1 \left(\frac{\partial f}{\partial u}\right)_0, \tag{19}$$

which can easily be verified by differentiation and use of Eq. (9). Equation (10) then becomes

$$\frac{d}{du_0}(v_1 - u_1 f_0) = (v_1 - u_1 f_0)^i \left(\frac{\partial f}{\partial v^i}\right)_0 + g_0, \tag{20}$$

which can also be derived from Eq. (18) by using Eq. (19) to eliminate G_0. For simplicity we write Eq. (20) as

$$\frac{d\tilde{v}_1}{du_0} = \tilde{v}_1{}^i \left(\frac{\partial f}{\partial v^i}\right)_0 + g_0, \tag{21}$$

where

$$\tilde{v}_1 = v_1 - u_1 f_0 . \tag{22}$$

Equation (21) is clearly identical in form to Eq. (15), which is the non-PL equation in first order. Thus, in solving Eq. (15) without the PL expansion, we are in effect solving Eq. (21) *with* the PL expansion, since u_1 has been incorporated into the unknowns $\tilde{v}_1$. A similar simplification can be effected in second order; we obtain

$$\frac{d\tilde{v}_2}{du_0} = \tilde{v}_2{}^i \left(\frac{\partial f}{\partial v^i}\right)_0 + \frac{1}{2}\tilde{v}_1{}^i \tilde{v}_1{}^j \left(\frac{\partial^2 f}{\partial v^i \partial v^j}\right)_0 + \tilde{v}_1{}^i \left(\frac{\partial g}{\partial v^i}\right)_0 + h_0, \tag{23}$$

where

$$\tilde{v}_2 = v_2 - u_2 f_0 - \frac{u_1{}^2}{2}\left[\left(\frac{\partial f}{\partial u}\right)_0 + f_0{}^i \left(\frac{\partial f}{\partial v^i}\right)_0\right] - u_1 \frac{d\tilde{v}_1}{du_0}, \tag{24}$$

as can be verified by differentiation and direct substitution. Equation (23) is identical in form to the non-PL equation (16), and the PL functions u_1 and u_2 appear in the new dependent variables $\tilde{v}_2$. Note that Eqs. (22) and (24) can be written in the alternative forms

$$\tilde{v}_1 = v_1 - u_1(dv_0/du_0), \tag{25}$$

$$\tilde{v}_2 = v_2 - u_2 \frac{dv_0}{du_0} - \frac{u_1^2}{2} \frac{d^2 v_0}{du_0^2} - u_1 \frac{d\tilde{v}_1}{du_0}, \tag{26}$$

as can be verified by inspection. Yet another form of Eq. (26) is found by using Eq. (21):

$$\tilde{v}_2 = v_2 - u_2 \frac{dv_0}{du_0} - \frac{u_1^2}{2} \frac{d^2 v_0}{du_0^2} - u_1 \left[\tilde{v}_1^i \left(\frac{\partial f}{\partial v^i} \right)_0 + g_0 \right]. \tag{27}$$

C. *Boundary Conditions*

To achieve complete correspondence between the solutions of Eqs. (10) and (21) and of Eqs. (11) and (23) we must examine the boundary conditions. It suffices for the applications of this paper to consider functions F in Eq. (1) that are regular within a finite domain about a boundary point

$$u = U \tag{28}$$

at which v is finite or zero:

$$v = V. \tag{29}$$

Other cases are discussed by Usher [8].

With reference to the perturbation equations (3) and (4) we take quite generally

$$v_0 = V_0, \tag{30}$$

$$v_i = V_i \qquad (i = 1, 2, \ldots), \tag{31}$$

such that

$$V = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots. \tag{31a}$$

Because the PL functions u_i $(i = 1, 2, \ldots)$ are arbitrary, their boundary values are also arbitrary. Let us choose

$$u_0 = U, \tag{32}$$

$$u_i = 0 \qquad (i = 1, 2, \ldots). \tag{33}$$

Therefore, if the conventional non-PL equations are used, boundary conditions (32) and (33) are in accord with Eqs. (12) and (13); i.e.,

$$u_i \equiv 0, \quad v_i(u) = V_i, \qquad (i = 1, 2, ...). \tag{34}$$

Concerning the boundary conditions on $\tilde{v}_1$ we see from Eqs. (22), (30)–(33) that

$$\tilde{v}_1(U) = V_1, \tag{35}$$

since by assumption f_0 is nonsingular at the boundary. A comparison of the tilde system of Eqs. (21) and (35) with the conventional non-PL system of Eqs. (15) and (34) reveals the interesting fact that they are identical except for the interpretation of the dependent and independent variables.

Concerning the boundary conditions on $\tilde{v}_2$, a similar argument using Eqs. (27), (30)–(33), and the fact that all functions and their derivatives are regular at the point (U, V), shows that $\tilde{v}_2(U) = 0$. Thus, to the order considered in this paper,

$$\tilde{v}_i(U) = V_i \qquad (i = 1, 2, ...). \tag{36}$$

D. *The New PL Approach*

We may summarize the results of the preceding sections as follows: When the point (U, V) is a regular point of the system (1), the perturbation equations (9), (21), and (23) with boundary conditions (30), (31), and (36) are identical to the non-PL equations (14)–(16) and (34), yet contain the PL feature through the new perturbation equations,

$$v = v_0 + \epsilon \left(\tilde{v}_1 + u_1 \frac{dv_0}{du_0}\right) + \epsilon^2 \left(\tilde{v}_2 + u_2 \frac{dv_0}{du_0} + \frac{u_1^2}{2} \frac{d^2 v_0}{du_0^2} + u_1 \frac{d\tilde{v}_1}{du_0}\right) + \cdots, \tag{37}$$

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots. \tag{38}$$

By inspection of the terms in parentheses we determine in each order the functions u_i $(i = 1, 2, ...)$ from the known functions v_0, $\tilde{v}_i$ $(i = 1, 2, ...)$ and their derivatives according to the particular needs of the problem. The non-PL expansion is recovered by choosing $u_i \equiv 0$, $(i > 0)$ in which case $\tilde{v}_i \equiv v_i$ $(i > 0)$.

We obtain the expansions (37) and (38) from Eqs. (3), (4), (22), and (26). The special case of Lindstedt's method [9] is obtained by setting, in Eqs. (37), (38),

$$u_i(u_0) = u_0 \gamma_i \qquad (i \to 1),$$

where γ_i = constant.

III. A Simple Derivation

Because the new PL expansions contain more terms than the conventional ones, they are more difficult to remember. However, a simple derivation of the new expansions exists which overcomes this difficulty and which also affords considerable insight into the new method.

In the conventional PL expansion

$$v = v_0(u_0) + \epsilon v_1(u_0) + \epsilon^2 v_2(u_0) + \cdots, \tag{39}$$

$$u = u_0 + \epsilon u_1(u_0) + \epsilon^2 u_2(u_0) + \cdots, \tag{40}$$

it has been *assumed* that the perturbation functions are all functions of u_0. Let us however expand v conventionally,

$$v = v_0(u) + \epsilon v_1(u) + \epsilon^2 v_2(u) + \cdots \tag{41}$$

and then apply Eq. (40). With Taylor's theorem it is not difficult to derive the transformed expansion for v which is identical to Eq. (37). It is reasonable to believe that the new PL expansion can be derived up to any order by this method, but we have not shown this rigorously. In addition the general validity of the new PL expansion must await development of the theory for arbitrary boundary conditions.

IV. A Simple Example

In studies of the one-zone approximation to a variable star, the following equations arise (Usher and Whitney [10]; Usher [11]) in the case of nonlinear adiabatic pulsations:

$$dx/dt = y, \tag{42}$$

$$dy/dt = -\omega^2 x + \epsilon\alpha_{20}x^2 + \epsilon^2\alpha_{30}x^3 + \cdots. \tag{43}$$

Here x is the amplitude, y the velocity, and ω the linear adiabatic frequency of the zone; α_{20} and α_{30} are constants and ϵ is a small parameter given by the ratio $r/r_0 = 1 + \epsilon x$ and whose magnitude is such that $x = O(1)$. In equilibrium the distance r from the center of the star is equal to r_0.

As initial conditions we choose

$$x = a_0 + \epsilon\left[\frac{a_0^2\alpha_{20}}{3\omega^2}\right] + \epsilon^2\left[\frac{a_0{}^3}{16\omega^2}\left(\frac{\alpha_{20}^2}{3\omega^2} - \frac{\alpha_{30}}{2}\right)\right] + \cdots, \tag{44}$$

$$y = 0 \tag{45}$$

at

$$t = 0, \tag{46}$$

where a_0 is some initial amplitude of the linear problem, i.e., we measure the time from a zero point in the velocity at which point the displacement is given by the value of Eq. (44). These initial conditions have been chosen in order to minimize the proliferation of terms as the solution is carried to successively higher orders. According to the prescription of Section II.D, we let

$$v = v_0 + \epsilon \left[\tilde{v}_1 + t_1 \frac{dv_0}{dt_0}\right] + \epsilon^2 \left[\tilde{v}_2 + t_2 \frac{dv_0}{dt_0} + \frac{t_1^2}{2} \frac{d^2 v_0}{dt_0^2} + t_1 \frac{d\tilde{v}_1}{du_0}\right] + \cdots, \tag{47}$$

$$t = t_0 + \epsilon t_1 + \epsilon^2 t_2 + \cdots, \tag{48}$$

where $v \equiv \{x, y\}$. Substituting (47) and (48) in (42) and (43) gives a system of equations identical (but for the zero subscripts on the new independent variable) to the system that would have resulted if no PL expansion had been used; i.e., the linearized equations.

$$dx_0/dt_0 = y_0\,, \tag{49}$$

$$dy_0/dt_0 = -\omega^2 x_0 \tag{50}$$

and in first order

$$d\tilde{x}_1/dt_0 = \tilde{y}_1\,, \tag{51}$$

$$d\tilde{y}_1/dt_0 = -\omega^2 \tilde{x}_1 + \alpha_{20} x_0^2, \tag{52}$$

while in second order we find

$$d\tilde{x}_2/dt_0 = \tilde{y}_2\,, \tag{53}$$

$$d\tilde{y}_2/dt_0 = -\omega^2 \tilde{x}_2 + 2\alpha_{20} x_0 \tilde{x}_1 + \alpha_{30} x_0^3. \tag{54}$$

By analogy with the choice of initial conditions in Section II.C in zero and higher orders, we have from equations (44)–(46):

$$x_0 = a_0\,, \qquad y_0 = 0, \quad t_0 = 0; \tag{55}$$

$$\tilde{x}_1 = a_0^2 \alpha_{20}/3\omega^2, \qquad \tilde{y}_1 = 0, \quad t_1 = 0; \tag{56}$$

$$\tilde{x}_2 = \frac{a_0^3}{16\omega^2}\left(\frac{\alpha_{20}^2}{3\omega^2} - \frac{\alpha_{30}}{2}\right), \quad \tilde{y}_2 = 0, \quad t_2 = 0. \tag{57}$$

The linear system describes a simple harmonic oscillator whose solution under conditions (55) is the generating solution

$$x_0 = a_0 \cos \omega t_0\,, \quad y_0 = -a_0 \omega \sin \omega t_0\,, \tag{58}$$

which possesses an irregular singularity at $t_0 = \infty$.

When the first-order equations (51) and (52) are considered along with the generating solution, it is not immediately obvious whether a PL expansion is necessary in order to ensure uniform convergence as $t \to \infty$ and, if so, what the form for t_1 should be. As a first step we solve Eqs. (51) and (52) to give

$$\tilde{x}_1 = -\frac{a_0{}^2\alpha_{20}}{2\omega^2}\left(1 + \frac{1}{3}\cos 2\omega t_0\right), \tag{59}$$

$$\tilde{y}_1 = \frac{a_0{}^2\alpha_{20}}{3\omega}\sin 2\omega t_0 . \tag{60}$$

It is immediately clear, without the necessity of considering the PL expansion, that no coordinate stretching is necessary in first order since there are no secular or secular periodic terms in the standard solution. Thus in the first-order terms in Eq. (47),

$$t_1 \equiv 0. \tag{61}$$

Consider next the second-order solution to Eqs. (53) and (54):

$$\tilde{x}_2 = \frac{a_0{}^3}{16\omega^2}\left(\frac{\alpha_{20}^2}{3\omega^2} - \frac{\alpha_{30}}{2}\right)\cos 3\omega t_0 + \frac{a_0{}^3}{2\omega}\left(\frac{3\alpha_{30}}{4} + \frac{5\alpha_{20}^2}{6\omega^2}\right) t_0 \sin \omega t_0; \tag{62}$$

$$\begin{aligned}\tilde{y}_2 = {} & \frac{a_0{}^3}{2\omega}\left(\frac{3\alpha_{30}}{4} + \frac{5\alpha_{20}}{6\omega^2}\right)\sin \omega t_0 - \frac{3a_0{}^3}{16\omega^2}\left(\frac{\alpha_{20}^2}{3\omega^2} - \frac{\alpha_{30}}{2}\right)\sin 3\omega t_0 \\ & + \frac{1}{2}a_0{}^3\left(\frac{3\alpha_{30}}{4} + \frac{5\alpha_{20}^2}{6\omega^2}\right) t_0 \cos \omega t_0 .\end{aligned} \tag{63}$$

Since these solutions diverge as $t_0 \to \infty$ we can use the PL function t_2 to attempt to remove the singularity. Since $t_1 \equiv 0$ from Eq. (61), we have from Eqs. (48) and the generating solution (58)

$$x_2 = \tilde{x}_2 - a_0\omega t_2 \sin \omega t_0 , \tag{64}$$

$$y_2 = \tilde{y}_2 - a_0\omega^2 t_2 \cos \omega t_0 . \tag{65}$$

To remove the secular periodic terms in the solutions (62) and (63) we let

$$t_2 = \frac{a_0{}^2}{2\omega^2}\left(\frac{3\alpha_{30}}{4} + \frac{5\alpha_{20}^2}{6\omega^2}\right), \tag{66}$$

which removes the singular terms from both x and y. We have therefore ascertained that a coordinate stretch is needed and have determined its magnitude without recourse to the particular solution of differential equations. Numerical verification of Eq. (66) is described elsewhere. (Usher [11].)

This elementary example illustrates the mechanics of the new PL approach to a particular class of problem and indicates that it will be more useful, the larger the order of the system. In the third-order system discussed elsewhere (Usher [11]), the new approach proved especially useful as a check of the solution which we had first found by conventional means. In fact, a discrepancy between the solutions by the two methods was traced to an algebraic error in the conventional method; that no errors were made in obtaining the solution by the new method may well be due to the more straightforward technique and ease of application of the new PL expansion.

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